1^{st}

International Mathematical Excellence Olympiad

Selected Problems with Solutions

2017

Mathematical Excellence

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Selected Problems with Solutions

This booklet is prepared by Amir Hossein Parvardi.
With special thanks to Konstantinos Metaxas and Amin Hashemi.

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Problem Proposers:

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Preface

The First International Mathematical Excalibur Olympiad was held on Spetember 15, 2017. This Olympiad is suitable for all the people who are interested in mathematics, especially high school students.

The competition consists of two levels. All students who are between 13 to 17 years old must participate in Level 1 and all students who are currently 17 years or older are supposed to participate in Level 2.

Each problem is worth 7 points. A partial solution to a problem is scored between 1 to 6 points, based on how complete it is. A solution without significant advancement will not get any points.

Students are supposed to start the solution for each problem in a new page in the following format: first write down the statement of the problem, and then the solution. Solutions must be presented in detail, including all necessary arguments and calculations. Providing all necessary figures of sufficient size is a must. If a problem has an explicit answer, this answer must be presented distinctly.

If a solution depends on some well-known theorems from standard textbooks, the participant may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

Any solution provided in this booklet is due to the proposer of the problem unless otherwise mentioned.

Problems

First Level

Problem 1. In a game, a player can level up to 16 levels. In each level, the player can upgrade an *ability* spending that level on it. There are three kinds of abilities, however, one ability can not be upgraded before level 6 for the first time. And that special ability can not be upgraded before level 11. Other abilities can be upgraded at any level, any times (possibly zero), but the special ability needs to be upgraded exactly twice. In how many ways can these abilities be upgraded?

Problem 2. Let O be the circumcenter of a triangle ABC. Let M be the midpoint of AO. The BO and CO intersect the altitude AD at points E and F, respectively. Let O_1 and O_2 be the circumcenters of the triangle ABE and ACF, respectively. Prove that M lies on O_1O_2.

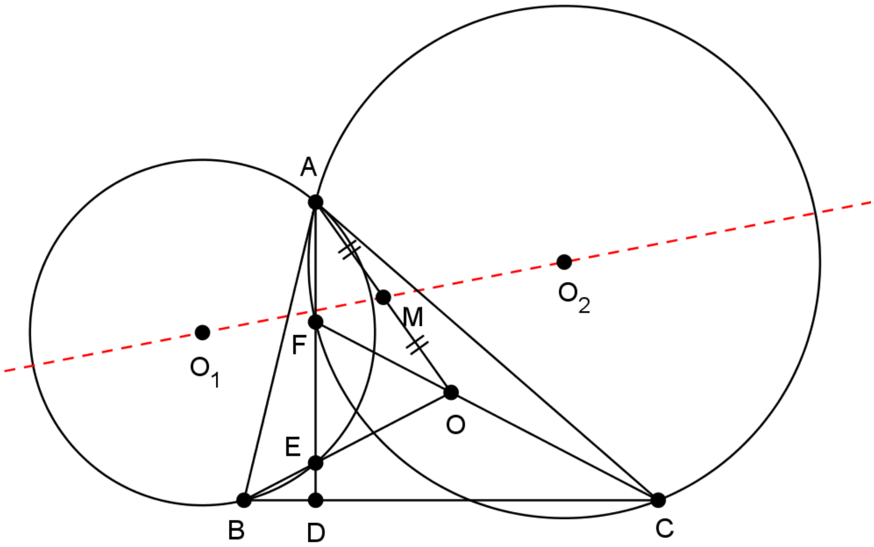


Figure 1: Figure of problem 2.

Problem 3. A triple (x, y, z) of real numbers is called a *superparticular* if

$$\frac{x+1}{x} \times \frac{y+1}{y} = \frac{z+1}{z},$$

Find all superparticular positive integer triples.

Problem 4. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\sqrt{\frac{a^3}{1+bc}} + \sqrt{\frac{b^3}{1+ac}} + \sqrt{\frac{c^3}{1+ab}} \geq 2.$$

Are there any triples (a, b, c) of real numbers for which the equality holds?

Second Level

Problem 1. Find all (or the least) positive integers n for which

$$(a^n - b^n)(b^n - c^n)(c^n - a^n)$$

is always divisible by 2^{2017} for any three odd positive integers a, b, and c.

Problem 2. Let A, B, C, and D be four points which lie on a plane , no three of them are collinear and the distance between any two points is a natural number. Prove that the distance between any two points is more than 1.

Problem 3. Prove that

$$\frac{a^4 + b^4}{b + c} + \frac{b^4 + c^4}{c + a} + \frac{c^4 + a^4}{a + b} \geq 3abc$$

For all positive reals a, b, and c.

Problem 4. Let L be a line and $\omega'(O)$ be a semicircle which intersects L at X and Y, as shown in Figure 1. Let $\omega(O')$ be a circle tangent to ω' and let Q be the perpendicular from O' to L. If ω and L are fixed and ω' is changeable then prove that for any constant β there is a point P on OQ such that $\angle XPY = \beta$.

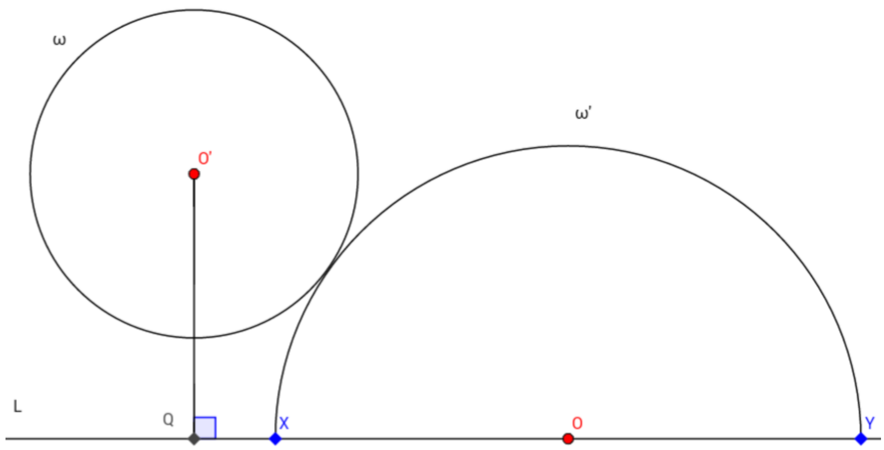


Figure 2: Figure of problem 4.

Solutions

First Level

Problem 1. In a game, a player can level up to 16 levels. In each level, the player can upgrade an *ability* spending that level on it. There are three kinds of abilities, however, one ability can not be upgraded before level 6 for the first time. And that special ability can not be upgraded before level 11. Other abilities can be upgraded at any level, any times (possibly zero), but the special ability needs to be upgraded exactly twice. In how many ways can these abilities be upgraded?

Proposed by Masum Billal

Solution. *First Solution.* Let us label the abilities as A, B, S with S denoting the special ability. Then any sequence of length 16 containing these characters is a way of leveling up all 16 levels if we assume that the appearing position of a character is the level it is being upgraded with. But in order for the sequence to be *valid*, we need the first position of S to be at least 6 and the second to be at least 11.
Since S has special focus, let us fix two position for S, u and v where $6 \leq u \leq v$ and $11 \leq v \leq 16$. From 1 to $u-1$, the number of ways A and B can appear depends on how many of which type is used. If we use type A for i times, then the other $u-1-i$ positions will be fixed for B and we can choose such positions for A in $\binom{u-1}{i}$ ways. Since i can vary from 0 to $u-1$, the number of ways to do this is $\sum_{i=0}^{u-1} \binom{u-1}{i} = 2^{u-1}$.
Now, in a similar manner, from position $u+1$ to $v-1$, the number of ways A and B can be placed is $2^{v-1-(u+1)+1} = 2^{v-1-u}$. And from $v+1$ to 16, the number of ways is again 2^{16-v}.
For the next portion, we will fix v only. For a fixed v, the total number of ways is:

$$\begin{aligned}
\sum_{u=6}^{v-1} 2^{u-1} \cdot 2^{v-1-u} \cdot 2^{16-v} &= \sum_{u=6}^{v-1} 2^{14} \\
&= 2^{14} \sum_{u=6}^{v-1} 1 \\
&= 2^{14} \cdot (v-6).
\end{aligned}$$

And, the total number of solutions is,

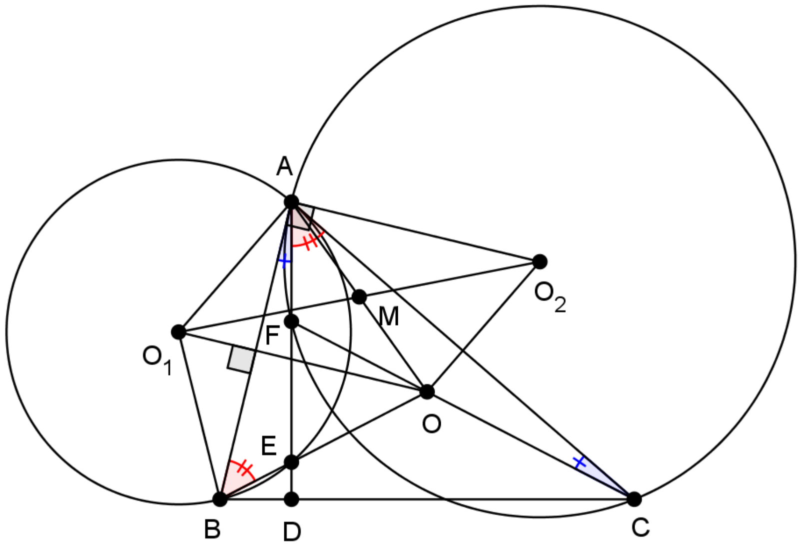
$$\sum_{v=11}^{16} 2^{14} \cdot (v-6) = 2^{14} \cdot 45.$$

Second Solution. This one is simpler. Fix two valid positions for S. Notice that, even if we change the positions for S, the number of ways to place A or B remains same. And if we simply put S in those positions, the number of ways A or B can be placed is the same as the number of binary strings possible with length $16-2=14$. Therefore, for each valid position of S, we get 2^{14} configurations and for symmetry, the solution is $K \cdot 2^{14}$ where K is the number of ways S can be assigned two valid positions. The rest is clear.

Problem 2. Let O be the circumcenter of a triangle ABC. Let M be the midpoint of AO. The BO and CO intersect the altitude AD at points E and F, respectively. Let O_1 and O_2 be the circumcenters of the triangle ABE and ACF, respectively. Prove that M lies on O_1O_2.

Proposed by Stefan Lozanovski

Solution. *First Solution.* Let ω_1 and ω_2 be the circumcircles of ABE and ACF, respectively.



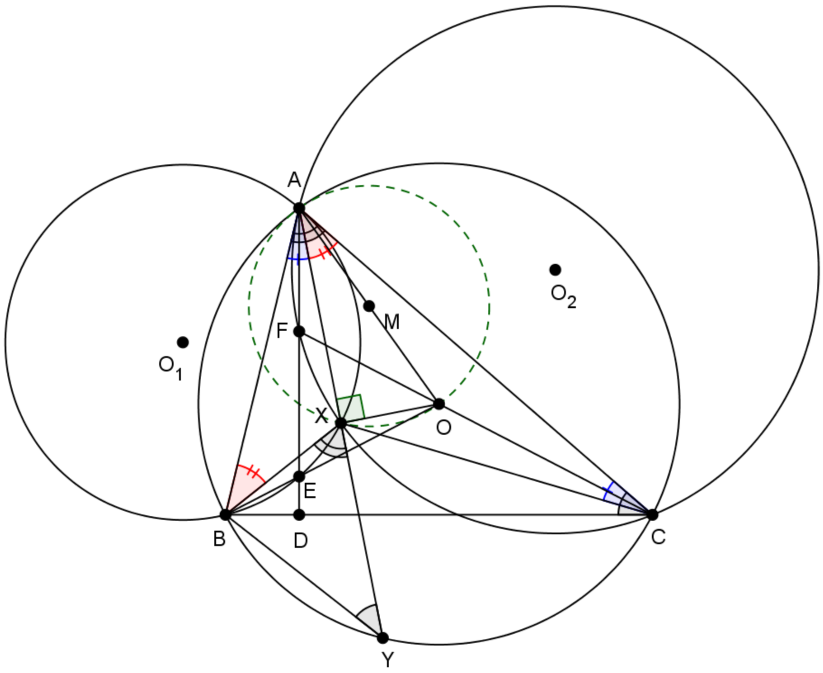
Then,

$$\begin{aligned}\angle ACF &=\equiv \angle ACO \\ &= \frac{180^\circ - \angle AOC}{2} = 90^\circ - \beta \\ &= 90^\circ - \angle ABD = \angle BAD \\ &=\equiv \angle BAF.\end{aligned}$$

Therefore, BA is tangent to ω_2, i.e., $BA \perp AO_2$. On the other hand, $\overline{O_1A} = \overline{O_1B}$ and $\overline{OA} = \overline{OB}$ as radii, so $O_1O \perp BA$. Therefore, $AO_2 \parallel O_1O$. Similarly, $AO_1 \parallel O_2O$. So, AO_1OO_2 is a parallelogram and because its diagonals bisect each other, $M \in O_1O_2$.

Second Solution. Let ω, ω_1 and ω_2 be the circumcircles of ABC, ABE and ACF, respectively. Let the second intersection of ω_1 and ω_2 be X.
Note that the points O_1 and O_2 and M are the centers of ω_1, ω_2 and the circle with diameter AO. It is sufficient to prove that these three circles are coaxial, i.e., that X lies on the circle with diameter AO. We need to prove that $\angle AXO = 90^\circ$, and because O is the circumcenter of ABC, we need to prove that X is the midpoint of the chord AY, where Y is the second intersection of AX and ω. Same as in first solution, we get that BA is tangent to ω_2 and CA is tangent to ω_1. From this, we get that $\angle XBA = \angle XAC$ and $\angle XAB = \angle XCA$, so $\triangle XBA \sim \triangle XAC$ and therefore

$$\frac{\overline{XB}}{\overline{XA}} = \frac{\overline{BA}}{\overline{AC}} \tag{1}$$



Also, $\angle BXY = \angle BAX + \angle XBA = \angle BAX + \angle XAC = \angle BAC$ and $\angle BYX \equiv \angle BYA = \angle BCA$, so $\triangle BXY \sim \triangle BAC$ and therefore

$$\frac{\overline{BX}}{\overline{XY}} = \frac{\overline{BA}}{\overline{AC}} \tag{2}$$

Finally, from 1 and 2, we can conclude that $\overline{AX} = \overline{XY}$.

Problem 3. A triple (x, y, z) of real numbers is called a *superparticular* if

$$\frac{x+1}{x} \times \frac{y+1}{y} = \frac{z+1}{z},$$

Find all superparticular positive integer triples.

Proposed by Amir Hossein Parvardi

Solution. Rewrite the equation as $z(x+1)(y+1) = xy(z+1)$, which simplifies to

$$z(x+y+1) = xy. \tag{3}$$

First, assume that $\gcd(x, y, z) = d$. Then there exist integers x_1, y_1, and z_1 with no common factor such that $x = dx_1, y = dy_1$, and $z = dz_1$. From these equations and equation (3), we get

$$dz_1(dx_1 + dy_1 + 1) = d^2x_1y_1, \text{ which implies } z_1(dx_1 + dy_1 + 1) = dx_1y_1.$$

So $d|z_1(dx_1+dy_1+1)$, which means $d|z_1$. Let $z_1 = dz_2$ and rewrite the above equation:

$$z_2(dx_1 + dy_1 + 1) = x_1y_1. \tag{4}$$

Clearly,

$$\gcd(x_1, y_1, z_2) = \gcd(x_1, y_1, z_1) = 1.$$

Now, suppose that $\gcd(x_1, z_2) = a$ and $\gcd(y_1, z_2) = b$. Clearly, $\gcd(a, b) = 1$ because otherwise x_1, y_1, and z_2 have a common factor. Since a and b are coprime integers either of which divides z_2, their product ab also divides z_2. So there exist positive integers x_2, y_2, and z_3 such that $x_1 = ax_2$, $y_1 = by_2$, and $z_2 = abz_3$. But $\gcd(z_3, x_2) = \gcd(z_3, y_2) = 1$. So by these equations and equation (4), we obtain

$$abz_3(dax_2 + dby_2 + 1) = abx_2y_2 \text{ which implies } z_3(dax_2 + dby_2 + 1) = x_2y_2. \quad (5)$$

From the final equation, we can see that $z_3|x_2y_2$, and since $\gcd(z_3, y_2) = 1$, we have $z_3|x_2$. But $\gcd(z_3, x_2) = 1$, which means that $z_3 = 1$. Therefore (5) becomes

$$dax_2 + dby_2 + 1 = x_2y_2. \quad (6)$$

Now rewrite (6) as

$$(x_2 - db)(y_2 - da) = d^2ab + 1. \quad (7)$$

Take $y_2 - da = t$. Then $y_2 = da + t$. Also $(x_2 - db)t = d^2ab + 1$, which gives

$$x_2 = \frac{d^2ab + dbt + 1}{t}.$$

Substituting the parameters, we finally find

$$(x, y, z) = \left(da \cdot \frac{d^2ab + dbt + 1}{t}, db(da + t), d^2ab\right),$$

where a, b, d, t are arbitrary positive integers such that $t|d^2ab + 1$ and $\gcd(a, b) = 1$.

Problem 4. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\sqrt{\frac{a^3}{1 + bc}} + \sqrt{\frac{b^3}{1 + ac}} + \sqrt{\frac{c^3}{1 + ab}} \geq 2.$$

Are there any triples (a, b, c) of real numbers for which the equality holds?

Proposed by Konstantinos Metaxas

Solution. Using the hypothesis and the the fact that $(a - 1)^2(a + 1) \geq 0 \Leftrightarrow a^3 + 1 \geq a^2 + a$, we obtain

$$\sum_{\text{cyc}} \sqrt{\frac{a^3}{1 + bc}} = \sum_{\text{cyc}} \sqrt{\frac{a^4}{a + 1}} = \sum_{\text{cyc}} \sqrt{\frac{a^5}{a^2 + a}} \geq \sum_{\text{cyc}} \sqrt{\frac{a^5}{a^3 + 1}} = \sum_{\text{cyc}} \frac{a^2}{\sqrt{a^2 + bc}}.$$

Using Cauchy–Schwarz inequality, we get

$$\sum_{\text{cyc}} \frac{a^2}{\sqrt{a^2 + bc}} \geq \frac{(a + b + c)^2}{\sqrt{a^2 + bc} + \sqrt{b^2 + ac} + \sqrt{c^2 + ab}}$$

Now we will prove the following lemma.

Lemma. *For non-negative a, b, and c we have*

$$3(a + b + c) \geq 2\left(\sqrt{a^2 + bc} + \sqrt{b^2 + ac} + \sqrt{c^2 + ab}\right).$$

Proof. Without loss of generality, we assume that $a \geq b \geq c$. Then,

$$\sqrt{b^2 + ca} + \sqrt{c^2 + ab} \leq \sqrt{2(b^2 + c^2) + 2a(b + c)}.$$

So, we need to prove that

$$2\sqrt{2(b^2 + c^2) + 2a(b + c)} + 2\sqrt{a^2 + bc} \leq 3(a + b + c).$$

Now let us set $u = \dfrac{1}{2}(b + c)$. Squaring the above inequality we get

$$8(b^2 + c^2 + 2au) \leq 9(a + 2u)^2 + 2(a^2 + bc) - 12(a + 2u)\sqrt{a^2 + bc},$$

which can be simplified to

$$(a - 2u)^2 + 20bc \geq 12(a + 2u)\left(\sqrt{a^2 + bc} - a\right).$$

Clearly,

$$\sqrt{a^2 + bc} - a = \frac{bc}{a + \sqrt{a^2 + bc}} \leq \frac{bc}{2a}.$$

This means it suffices to prove that

$$(a - 2u)^2 + 20bc \geq \frac{6bc(a + 2u)}{a},$$

which can be written as

$$(a - 2u)^2 + 2bc + \frac{12bc(a - u)}{a} \geq 0.$$

The last inequality holds because $a \geq u$. Equality holds if and only if $a = b$ and $c = 0$ (and permutations, of course). $\square$

Now using the above lemma we get that

$$\frac{(a + b + c)^2}{\sqrt{a^2 + bc} + \sqrt{b^2 + ac} + \sqrt{c^2 + ab}} \geq \frac{2(a + b + c)}{3} \geq 2,$$

where the last inequality follows from AM-GM. The inequality is thus proven.
Clearly, there are no triples (a, b, c) for which the equality occurs, since the equality in

$$\frac{(a + b + c)^2}{\sqrt{a^2 + bc} + \sqrt{b^2 + ac} + \sqrt{c^2 + ab}} \geq \frac{2(a + b + c)}{3}$$

holds only if $c = 0$ and $a = b$ (and permutations).

Second Level

Problem 1. Find all (or the least) positive integers n for which

$$(a^n - b^n)(b^n - c^n)(c^n - a^n)$$

is always divisible by 2^{2017} for any three odd positive integers a, b, and c.

Proposed by Tynyshbek Anuarbekov

Solution. Answer: $n = 2^{670}k$ for all $k \in \mathbb{N}$.
First we prove that n must be divisible by 2^{670}. Note that one of $a^n - b^n, b^n - c^n$, or $c^n - a^n$ must be divisible by 2^{673} because otherwise

$$v_2\left(\left(a^n - b^n\right)\left(b^n - c^n\right)\left(c^n - a^n\right)\right) \leq 3 \cdot 672 = 2016,$$

which is a contradiction.
We take $\{a, b, c\} = \{1, 3, 5\}$. Without loss of generality, assume that $2^{673} \mid a^n - b^n$. Then,

$$a^n \equiv b^n \pmod{2^{673}}$$

Let d be the least positive integer x (order of an element for two numbers) for which

$$a^x \equiv b^x \pmod{2^{673}}.$$

It is not hard to prove that $a^{2^{671}} \equiv b^{2^{671}} \pmod{2^{673}}$ for odd positive integers a and b. We know that $d \mid 2^{671}$. Therefore, d must be a power of 2. Let $d = 2^t$ for some integer t. Assume that $t \leq 669$. Then we have $a^{2^{669}} \equiv b^{2^{669}} \pmod{2^{673}}$ which is not true because by LTE we have

$$v_2\left(a^{2^{669}} - b^{2^{669}}\right) = v_2\left(a^2 - b^2\right) + 668 \leq 672$$

for $\{a, b\} \in \{1, 3, 5\}$. It means that $t \geq 670$ and since $d \mid n$ so we have proven that $2^{670} \mid n$.
Now we prove that if $n = 2^{670}k$ for any $k \in \mathbb{N}$, then

$$\left(a^n - b^n\right)\left(b^n - c^n\right)\left(c^n - a^n\right)$$

is divisible by 2^{2017} for all odd positive integers a, b, and c. It is easy to see that we only need to prove that

$$2^{2017} \mid (a^{2^{670}} - b^{2^{670}})(b^{2^{670}} - c^{2^{670}})(c^{2^{670}} - a^{2^{670}}).$$

We prove a lemma first.

Lemma. *If a, b, c are odd then $\left(a^2 - b^2\right)\left(b^2 - c^2\right)\left(c^2 - a^2\right)$ is divisible by 2^{10}.*

Proof. We already know from elementary number theory that for any two odd positive integers x and y, we have $8 \mid x^2 - y^2$. So, there exist integers k, l, and m such that

$$\begin{aligned} a^2 - b^2 &= 8k, \\ b^2 - c^2 &= 8l, \\ c^2 - a^2 &= 8m. \end{aligned}$$

Summing up these three equations gives us $k + l + m = 0$, which means at least one of k, l, or m must be ever. Without loss of generality, suppose that $k = 2s$ for some integer s. Then,

$$\left(a^2 - b^2\right)\left(b^2 - c^2\right)\left(c^2 - a^2\right) = 2^{10} slm,$$

which is clearly divisible by 2^{10}. □

From the above lemma we obtain

$$v_2\left(\left(a^2 - b^2\right)\left(b^2 - c^2\right)\left(c^2 - a^2\right)\right) = v_2\left(a^2 - b^2\right) + \ v_2\left(b^2 - c^2\right) + \ v_2\left(a^2 - c^2\right) \geq 10.$$

Now, by LTE, we have

$$v_2\left(a^{2^{670}} - b^{2^{670}}\right) = v_2\left(a^2 - b^2\right) + 669,$$

and so,

$$\begin{aligned} v_2\left(\left(a^{2^{670}} - b^{2^{670}}\right)\left(b^{2^{670}} - c^{2^{670}}\right)\left(c^{2^{670}} - a^{2^{670}}\right)\right) &= v_2\left(a^2 - b^2\right) + v_2\left(b^2 - c^2\right) + v_2\left(a^2 - c^2\right) \\ &\quad + 3 \cdot 669 \\ &\geq 10 + 2007 \\ &= 2017. \end{aligned}$$

Problem 2. Let A, B, C, and D be four points which lie on a plane, no three of them are collinear and the distance between any two points is a natural number. Prove that the distance between any two points is more than 1.

Proposed by Zurab Agdgomelashvili

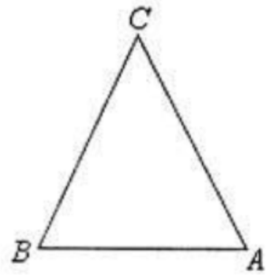
Solution. Geometric figures of whole numbers and their properties in the topic, that is a bridge between the questions of elementary mathematic and serious mathematics. In general, we name Diophantine polygons (tetrahedrons) the polygons (tetrahedrons), in which the distance between any two vertices is a natural number.
First we shall prove the following two Lemmas.

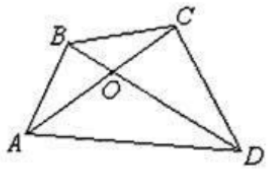
Lemma (Lemma 1)**.** *If the length of a Diophantine triangle's side is equal to* 1*, then the two remaining sides are equal.*

Proof. Let the triangle be ABC, as in the figure below. Without loss of generality, we can assume that $|AB| = 1$ and that $|AB| \geq |BC|$. Using triangle inequality, one can obtain

$$|AB| + |BC| > |AC| \implies 1 + |BC| > |AC|.$$

Now, since both $|AC|$ and $|BC|$ are positive integers, we must have $|BC| \geq |AC|$, which immediately results in $|AC| = |BC|$, as desired. □





Lemma (Lemma 2)**.** *The length of every convex Diophantine quadrilateral's side and diagonal is greater than* 1*.*

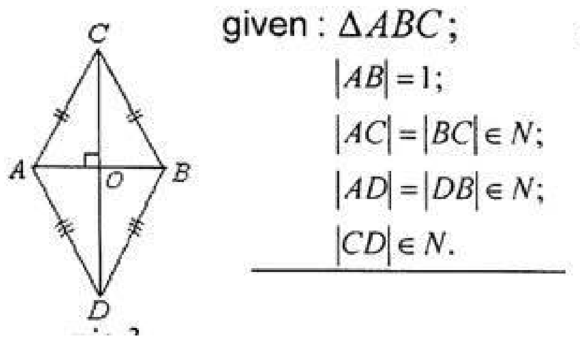
Proof. Let us assume a quadrilateral $ABCD$ as in figure below. At first we will prove that each side is greater than 1. For the sake of contradiction, let us assume, without loss of generality, that $|AB| = 1$. Then, using Lemma , we get $|BC| = |AC|$ and $|BD| = |AD|$. But this impossible because B and C must be in the middle of $[AB]$. So, we proved that each side is greater than 1. Now, we will prove that each diagonal is greater than 1. Suppose for the sake of contradiction that $|AC| = 1$. Again, by Lemma and triangle inequality in triangles BOC and AOD, we easily find that

$$|BD| + |AC| > |BC| + |AD| \implies |BD| + 1 > |AB| + |AD|.$$

Since all values are integers, we have $|BD| \geq |AB| + |AD|$. This is impossible because triangle inequality in ABD gives $|AB| + |AD| > |BD|$. So, we have a contradiction and the proof is complete. □

Now we will prove our problem. For the sake of contradiction let us assume that there exist points, no three of which are co-linear, their distances are natural numbers and, one distance is equal to 1. Then, by Lemma the remaining two points must lie in the perpendicular bisector of the side whose length is 1. Now we have two cases.

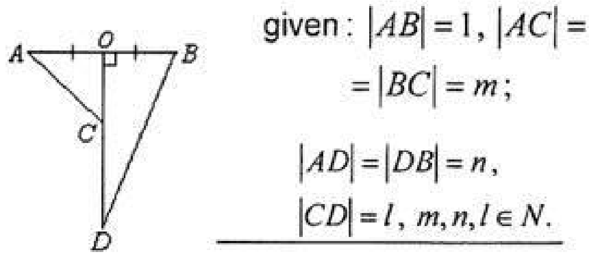
- First case: As shown in the figure,



$$|AB| = 1, |AC| = |BC|, \text{and } |AD| = |DB|.$$

All side-lengths are natural numbers. This case is discussed in Lemma .

- Second case:



As shown in the figure above,

$$|AB| = 1, |AC| = |BC| = m, |AD| = |DB| = n, \text{and } |CD| = l.$$

For some natural numbers l, m, and n. Now, from the right-angled triangles AOC and DOB we get $|AC|^2 = |AO|^2 + |OC|^2$ and $|BD|^2 = |OB|^2 + |OD|^2$. Moreover,

$$\begin{aligned}
|OD| &= |OC| + |CD|, \\
|AO| &= |OB| = \frac{1}{2}, \\
|AC| &= |CB| = m, \\
|AD| &= |DB| = n, \\
|CD| &= l.
\end{aligned}$$

From the above relations we get

$$m^2 = 1/4 + |OC|^2, \tag{8}$$

and

$$n^2 = 1/4 + (l + |OC|)^2. \tag{9}$$

Equation 8 can be written as

$$4m^2 - 1 = 4|OC|^2. \tag{10}$$

Subtracting Equation 8 from Equation 9, we find

$$2|OC| = \frac{n^2 - m^2 - l^2}{l} = q, \tag{11}$$

for some positive rational q. Finally, 11 becomes

$$4m^2 - 1 = q^2.$$

Since $4m^2 - 1$ is a natural number, so is q^2, and since q is a rational number whose square is an integer, q itself must be an integer. Therefore, we can write the last equation as

$$(2m - q)(2m + q) = 1.$$

This means that $2m - q = 2m + q = 1$, which is impossible.

We have investigated both cases and the proof is complete.

Problem 3. Prove that

$$\frac{a^4+b^4}{b+c}+\frac{b^4+c^4}{c+a}+\frac{c^4+a^4}{a+b}\geq 3abc$$

For all positive reals a, b, and c.

Proposed by Stefan Lozanovski

Solution. *First Solution.* Without loss of generality, suppose that $a \geq b$. This implies $a^3 \geq b^3$. By Chebyshev's inequality, we have

$$\begin{aligned} a^4+b^4 &= a\cdot a^3+b\cdot b^3 \\ &\geq \frac{1}{2}(a+b)(a^3+b^3). \end{aligned}$$

One can similarly prove the following two inequalities as well:

$$\begin{aligned} b^4+c^4 &\geq \frac{1}{2}(b+c)(b^3+c^3), \\ c^4+a^4 &\geq \frac{1}{2}(c+a)(c^3+a^3). \end{aligned}$$

Now,

$$\sum_{\text{cyc}}\frac{a^4+b^4}{b+c}\geq \sum_{\text{cyc}}\frac{(a+b)(a^3+b^3)}{2(b+c)}.$$

By AM-GM, we have

$$\begin{aligned} \sum_{\text{cyc}}\frac{(a+b)(a^3+b^3)}{2(b+c)} &\geq \frac{3}{2}\prod_{\text{cyc}}\left(\sqrt[3]{a^3+b^3}\cdot\sqrt[3]{\frac{a+b}{b+c}}\right) \\ &= \frac{3}{2}\sqrt[3]{\prod_{\text{cyc}}(a^3+b^3)\cdot 1}. \end{aligned}$$

Using AM-GM again,

$$\begin{aligned} \frac{3}{2}\sqrt[3]{\prod_{\text{cyc}}(a^3+b^3)\cdot 1} &\geq \frac{3}{2}\sqrt[3]{\prod_{\text{cyc}}(2\sqrt{a^3b^3})} \\ &= \frac{3}{2}\sqrt[3]{2^3a^3b^3c^3} \\ &= \frac{3}{2}2abc \\ &= 3abc. \end{aligned}$$

The proof is finished now.

Second Solution (by Dimitar Trenevski). Notice that

$$\sum_{\text{cyc}} \frac{a^4+b^4}{b+c} = \sum_{\text{cyc}} \frac{a^4}{b+c} + \sum_{\text{cyc}} \frac{b^4}{b+c}$$
$$= \frac{\displaystyle\sum_{\text{cyc}} \frac{a^4}{b+c} \sum_{\text{cyc}} (b+c)}{\displaystyle 2\sum_{\text{cyc}} a} + \frac{\displaystyle\sum_{\text{cyc}} \frac{b^4}{b+c} \sum_{\text{cyc}} (b+c)}{\displaystyle 2\sum_{\text{cyc}} a}.$$

By Cauchy-Schwarz inequality,

$$\frac{\displaystyle\sum_{\text{cyc}} \frac{a^4}{b+c} \sum_{\text{cyc}} (b+c)}{\displaystyle 2\sum_{\text{cyc}} a} + \frac{\displaystyle\sum_{\text{cyc}} \frac{b^4}{b+c} \sum_{\text{cyc}} (b+c)}{\displaystyle 2\sum_{\text{cyc}} a} \geq \frac{\left(\displaystyle\sum_{\text{cyc}} a^2\right)^2}{\displaystyle 2\sum_{\text{cyc}} a} + \frac{\left(\displaystyle\sum_{\text{cyc}} b^2\right)^2}{\displaystyle 2\sum_{\text{cyc}} a}$$
$$= \frac{2\left(\displaystyle\sum_{\text{cyc}} a^2\right)^2}{\displaystyle 2\sum_{\text{cyc}} a}$$
$$= \frac{\displaystyle\sum_{\text{cyc}} a^2 \sum_{\text{cyc}} a^2}{\displaystyle\sum_{\text{cyc}} a}.$$

By Chebyshev's inequality,

$$\frac{\displaystyle\sum_{\text{cyc}} a^2 \sum_{\text{cyc}} a^2}{\displaystyle\sum_{\text{cyc}} a} \geq \frac{\left(\displaystyle\sum_{\text{cyc}} a\right)^4}{\displaystyle 3^2\sum_{\text{cyc}} a}$$
$$= \frac{\left(\displaystyle\sum_{\text{cyc}} a\right)^3}{3^2}.$$

Finally, by AM-GM,

$$\frac{\left(\displaystyle\sum_{\text{cyc}} a\right)^3}{3^2} \geq \frac{(3\sqrt[3]{abc})^3}{3^2}$$
$$= 3abc,$$

which is what we were looking for.

Problem 4. Let L be a line and $\omega'(O)$ be a semicircle which intersects L at X and Y, as shown in Figure 1. Let $\omega(O')$ be a circle tangent to ω' and let Q be the perpendicular from O' to L. If ω and L are fixed and ω' is changeable then prove that for any constant β there is a point P on OQ such that $\angle XPY = \beta$.

Proposed by Amin Hashemi

Solution. The solution is due Mihai Theodor Iliant from Romania.
The constant β belongs to the interval $[0, \pi]$ (considering modulo π). I am going to show that $(0, \pi)$ is attainable ($0, \pi$ being trivial).
Let $O(t)$ be the center of the circle that varies, and impose the condition that is lies on $O'Q$, beyond Q straight downwards.

$$OQ' = a,$$
$$QO(t) = b(t).$$

Of course, $b(t)$ tends to infinity as $O(t)$ points towards that "downwards infinity". We have

$$\angle X(t)O(t)Y(t) = 2 \cdot \angle Q(t)O(t)Y(t) = 2 \cdot c(t).$$

I will show that $c(t)$ tends to 0, the previous being the aforementioned limit.

$$\cos\big(c(t)\big) = \frac{QO(t)}{Y(t)O(t)} = \frac{b(t)}{a + b(t)} = \frac{1}{1 + \frac{a}{b(t)}},$$

and the latter tends to 1, i.e. $\cos\big(c(t)\big)$ tends to 1, and thus $c(t)$ tends to 0. We can achieve any sufficiently small angle due to this, and any angle bigger that it as well(from it to π), by moving $P(t)$ accordingly.