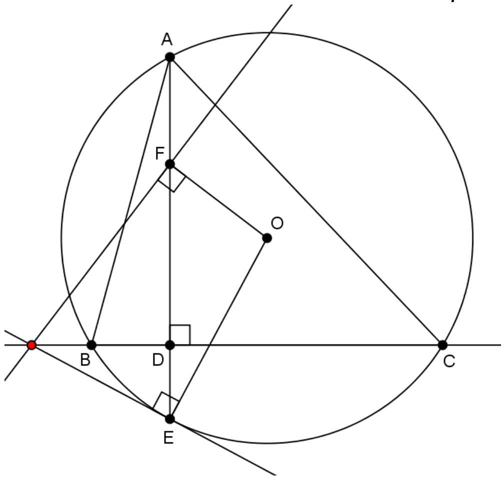
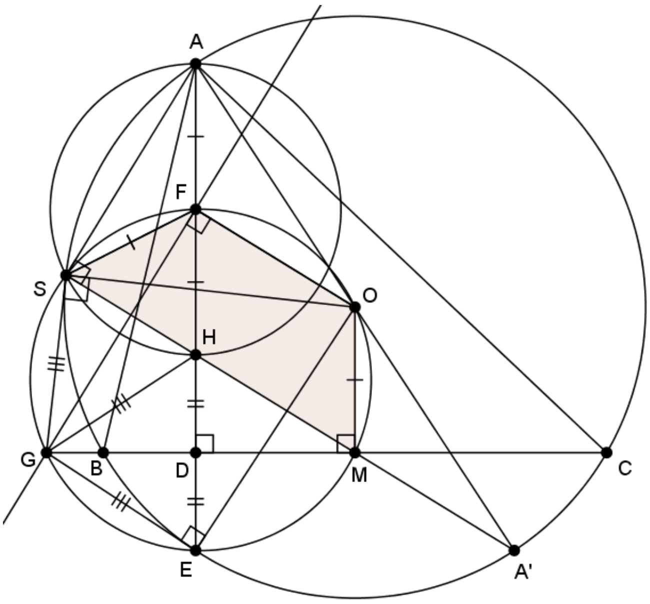
Problem. Let ABC be an acute triangle with circumcircle ω and circumcenter O. The perpendicular from A to BC intersects BC and ω at D and E, respectively. Let F be a point on the segment AE, such that $2 \cdot \overline{FD} = \overline{AE}$. Let ℓ be the perpendicular to OF through F. Prove that ℓ, the tangent to ω at E and the line BC are concurrent.

Proposed by: Stefan Lozanovski, MKD



Proof (Stefan Lozanovski).
Let $\ell \cap BC = G$. We will prove that GE is tangent to ω. Let H be the orthocenter of ABC. It is well-known that $\overline{HD} = \overline{DE}$. From $2 \cdot \overline{FD} = \overline{AE}$ we get that F is the midpoint of AH.



Let M be the midpoint of BC. It is well known that MH passes through A' - the antipode of A in ω. If S is the second intersection of MH and ω, then $\angle HSA \equiv \angle A'SA = 90°$, so S lies on the circle with diameter AH, which is centered at F. Therefore $\overline{FS} = \overline{FH}$ ... (1)

Since $\angle MSA \equiv \angle A'SA = 90°$ we have $MH \perp AS$. But since O and F are centers of (ABC) and (ASH) and AS is their common chord, we have $OF \perp AS$. Therefore, $MH \parallel OF$ ... (2)

Since FH and OM are both perpendicular to BC, we get $FH \parallel OM$. Using (2), we get that $OFHM$ is a parallelogram. Therefore $\overline{FH} = \overline{OM}$. (This can alternatively be proven by the well-known fact that $\overline{AH} = 2 \cdot \overline{OM}$). Using (1), we get that $\overline{FS} = \overline{OM}$. Using (2) again, we get that $OFSM$ is an isosceles trapezoid and therefore it's cyclic. Using that $OFGM$ is also cyclic ($\angle OFG = 90° = \angle OMG$), we get that $OFSGM$ is cyclic and therefore $\angle OSG = \angle OFG = 90°$.

We have $HS \parallel OF$ and $OF \perp FG$, so $HS \perp FG$. Using (1), we get that FG is the side bisector of SH, so $\overline{GS} = \overline{GH}$. Since $\overline{HD} = \overline{DE}$, we also get $\overline{GH} = \overline{GE}$. Therefore $\overline{GS} = \overline{GH} = \overline{GE}$.

Finally, we get that $\triangle OSG \cong \triangle OEG$ (by SSS), so $\angle OEG = \angle OSG = 90°$, i.e. GE is tangent to ω ■