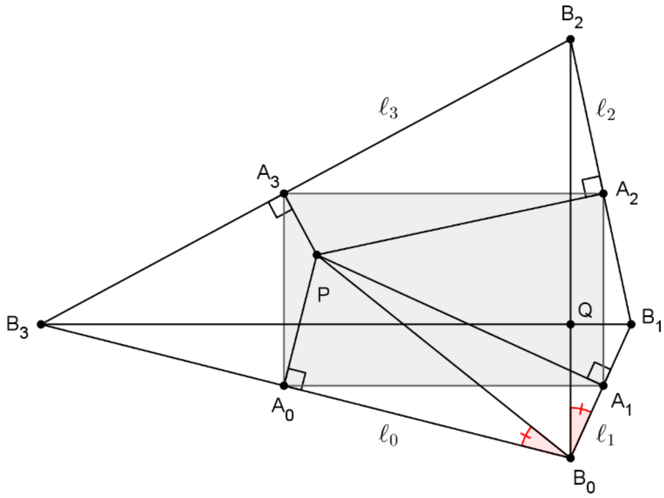
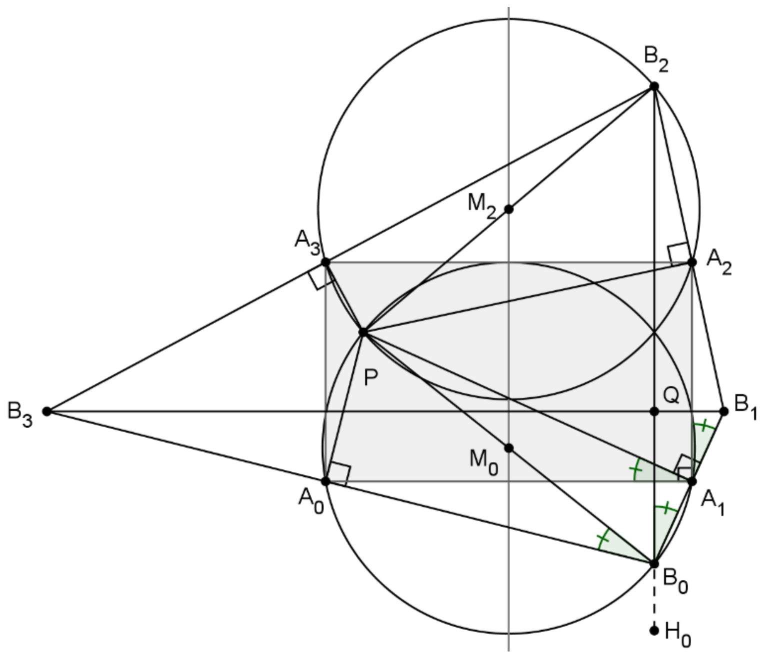
Problem. Let P be a point inside the rectangle $A_0A_1A_2A_3$. Let ℓ_i be the line through A_i perpendicular to PA_i and let $\ell_i \cap \ell_{i+1} = B_i$. Let Q be the intersection of the diagonals in $B_0B_1B_2B_3$. Prove that $\angle PB_iB_{i-1} = \angle QB_iB_{i+1}$. (All notations are for all $i \in \{0,1,2,3\}$ and are considered $mod\ 4$)

Proposed by: Stefan Lozanovski, MKD



Proof (Stefan Lozanovski).
Since $\angle PA_0B_0 + \angle PA_1B_0 = 90° + 90° = 180°$, we get $PA_0B_0A_1$ is a cyclic quadrilateral with center at M_0 - the midpoint of the diameter PB_0.



Since the center of a circle lies on the side bisectors of its chords, we get that M_0 lies on the side bisector of A_0A_1 and similarly, M_2 – the midpoint of PB_2 lies on the side bisector of A_2A_3. Since $A_0A_1A_2A_3$ is a rectangle, the opposite sides share a common side bisector, which is parallel to the other pair of opposite sides. Therefore, $M_0M_2 \parallel A_0A_3 \parallel A_1A_2$.

On the other hand, since M_0M_2 is midsegment in $\triangle PB_0B_2$, we get that $M_0M_2 \parallel B_0B_2$.

Therefore, $B_0B_2 \parallel A_0A_3 \parallel A_1A_2$ and similarly $B_1B_3 \parallel A_0A_1 \parallel A_2A_3$.

Now, there are two ways to finish the problem:
(We will prove the assertion only for $i = 0$, the other cases are analogous)

<u>I way (angle chasing):</u>
1 - $PA_0B_0A_1$ is cyclic
2 - $\angle A_0A_1A_2 = 90°$ (rectangle $A_0A_1A_2A_3$)
3 - $\angle PA_1B_1 = 90°$ ($\ell_1 \perp PA_1$)
4 - $A_2A_1 \parallel B_2B_0$

$\angle PB_0B_3 \equiv \angle PB_0A_0 \overset{1}{=} \angle PA_1A_0 \overset{2}{=} 90° - \angle PA_1A_2 \overset{3}{=} \angle A_2A_1B_1 \overset{4}{=} \angle B_2B_0B_1 \equiv \angle QB_0B_1$ ■

<u>II way (circumcenter and orthocenter are isogonal conjugates):</u>
Let H_0 be the orthocenter of $\triangle A_0B_0A_1$. Since $B_0B_2 \parallel A_1A_2$ and $A_1A_2 \perp A_0A_1$, we get $B_0B_2 \perp A_0A_1$, i.e. $H_0 \in B_0B_2 \equiv B_0Q$. We also know that M_0 (the circumcenter of $\triangle A_0B_0A_1$) lies on B_0P. It is well-known that the orthocenter and the circumcenter are isogonal conjugates in a triangle, so we get that B_0P and B_0Q are isogonal lines w.r.t $B_0A_0 \equiv B_0B_3$ and $B_0A_1 \equiv B_0B_1$ ■