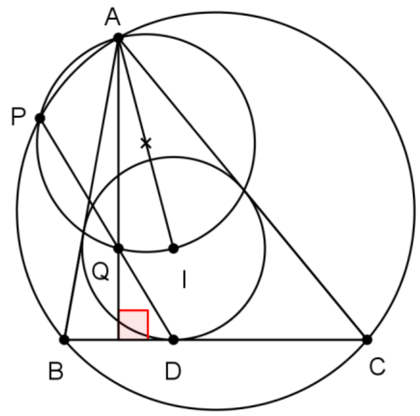
Problem. Let ABC be a triangle with circumcircle ω. Let I be its incenter and let D be the tangent point of the incircle and the side BC. Let the circle with diameter AI intersect ω at $P \neq A$ and PD at $Q \neq P$. Prove that $AQ \perp BC$.

Proposed by: Nikola Danevski and Stefan Lozanovski, Macedonia



Proof (Nikola Danevski and Stefan Lozanovski)**.**

The proof is written backwards, in order to reflect the motivation the student needs for each step.

Since $AQ \perp QI$, we need to prove that $QI \parallel BC$. We want to prove that $\angle IQD = \angle QDB$, but since $IQPA$ is cyclic, we know that $\angle IQD = \angle PAI$, so we have to prove that $\angle PAI = \angle QDB \equiv \angle PDB$, i.e. we have to prove that $APDS$ is cyclic, where $S = AI \cap BC$. For that, we need $\angle APD = \angle ASC$. But $\angle ASC = \beta + \frac{\alpha}{2}$ (as exterior angle in $\triangle ABS$) and $\angle APD = \angle APC + \angle CPD = \beta + \angle CPD$, so we need to prove that $\angle CPD = \frac{\alpha}{2}$. Since $\angle BPC = \angle BAC = \alpha$, we need to prove that PD is the angle bisector of $\angle BPC$. We will prove that by the angle bisector theorem in $\triangle PBC$.

Let E and F be the tangent points of the incircle and the sides CA and AB, respectively. Obviously, they lie on the circle with diameter AI. We now focus on the triangles PBF and PCE.

$$\angle PBF \equiv \angle PBA = \angle PCA \equiv \angle PCE$$

$$\angle PFB = 180° - \angle PFA = 180° - \angle PEA = \angle PEC$$

Therefore, by the AA criterion, $\triangle PBF \sim \triangle PCE$ and therefore

$$\frac{\overline{PB}}{\overline{BF}} = \frac{\overline{PC}}{\overline{CE}}.$$

Finally, since $\overline{BF} = \overline{BD}$ and $\overline{CE} = \overline{CD}$ as tangent segments, we get

$$\frac{\overline{PB}}{\overline{BD}} = \frac{\overline{PC}}{\overline{CD}},$$

i.e. PD is angle bisector in $\triangle BPC$ ■