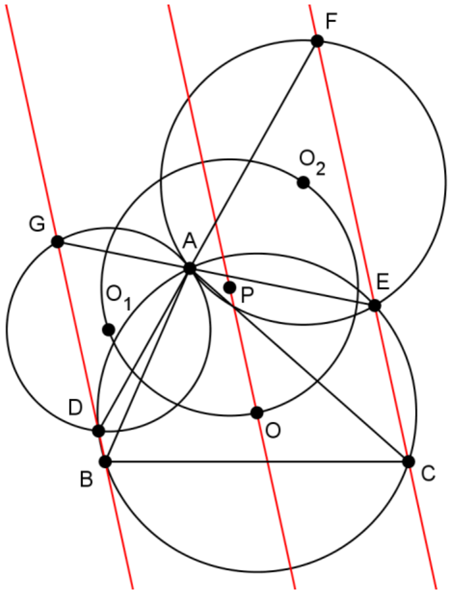
Problem. Let ABC be a triangle with circumcircle ω. Let ℓ_B and ℓ_C be two lines though the points B and C, respectively, such that $\ell_B \parallel \ell_C$. The second intersections of ℓ_B and ℓ_C with ω are D and E, respectively ($D \in \widehat{AB}, E \in \widehat{AC}$). Let DA intersect ℓ_C at F and let EA intersect ℓ_B at G. If O, O_1 and O_2 are circumcenters of ABC, ADG and AEF, respectively, and P is the circumcenter of OO_1O_2, prove that $\ell_B \parallel OP \parallel \ell_C$.

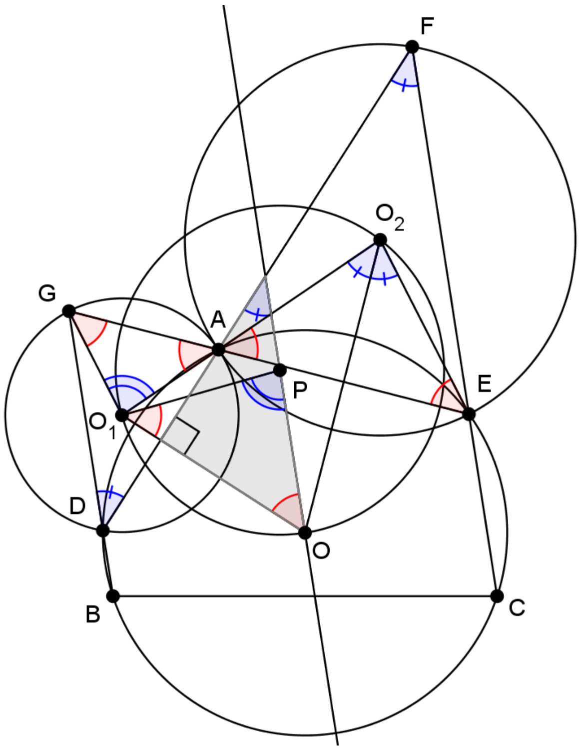
Proposed by: Stefan Lozanovski, Macedonia



Proof (Stefan Lozanovski).
Since O_1, O_2 are circumcenters in $(AGD), (AEF)$ and since $GD \parallel FE$, we get

$$\angle GAO_1 = 90° - \frac{\angle GO_1A}{2} = 90° - \angle GDA = 90° - \angle EFA = 90° - \frac{\angle EO_2A}{2} = \angle EAO_2,$$

which means that $A \in O_1O_2$.



Let $\angle(DF, FE) = \varphi$. Then, as central angle $\angle AO_2E = 2\varphi$. Since AE is the radical axis of ω and ω_2, OO_2 bisects $\angle AO_2E$, i.e. $\angle AO_2O = \varphi$. Since $A \in O_1O_2$, we get $\angle O_1O_2O \equiv \angle AO_2O = \varphi$. Since P is the circumcenter of (O_1O_2O), we get $\angle O_1PO = 2\varphi$ and $\angle POO_1 = 90° - \varphi$. Since AD is radical axis of ω and ω_1, we get $OO_1 \perp DA$. Since $\angle(DF, OO_1) = 90°$ and $\angle(PO, OO_1) = 90° - \varphi$, we get that $\angle(DF, PO) = \varphi = \angle(DF, FE)$, i.e. $PO \parallel FE$ ■