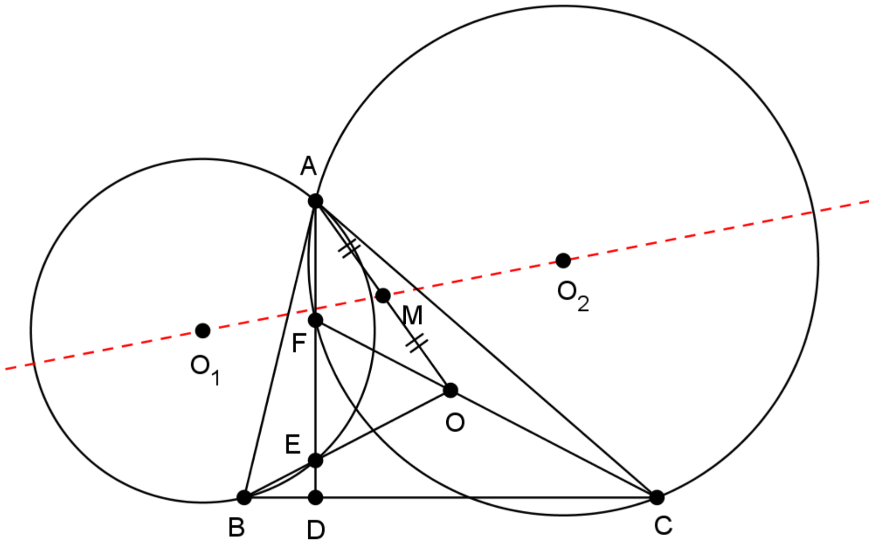
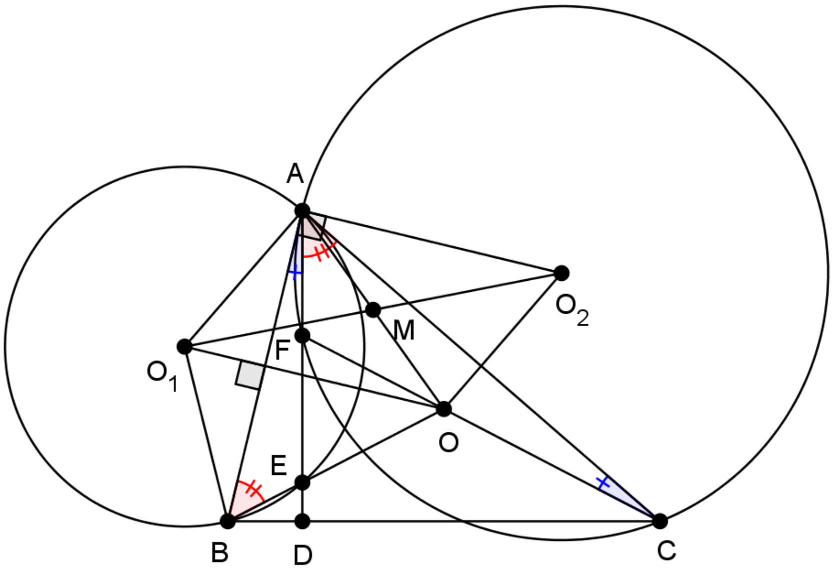
**Problem.** Let $O$ be the circumcenter of a triangle $ABC$. Let $M$ be the midpoint of $AO$. The lines $BO$ and $CO$ intersect the altitude $AD$ at points $E$ and $F$, respectively. Let $O_1$ and $O_2$ be the circumcenters of the triangles $ABE$ and $ACF$, respectively. Prove that $M$ lies on $O_1O_2$.

*Proposed by: Stefan Lozanovski, Macedonia*




**Proof 1** (Stefan Lozanovski).

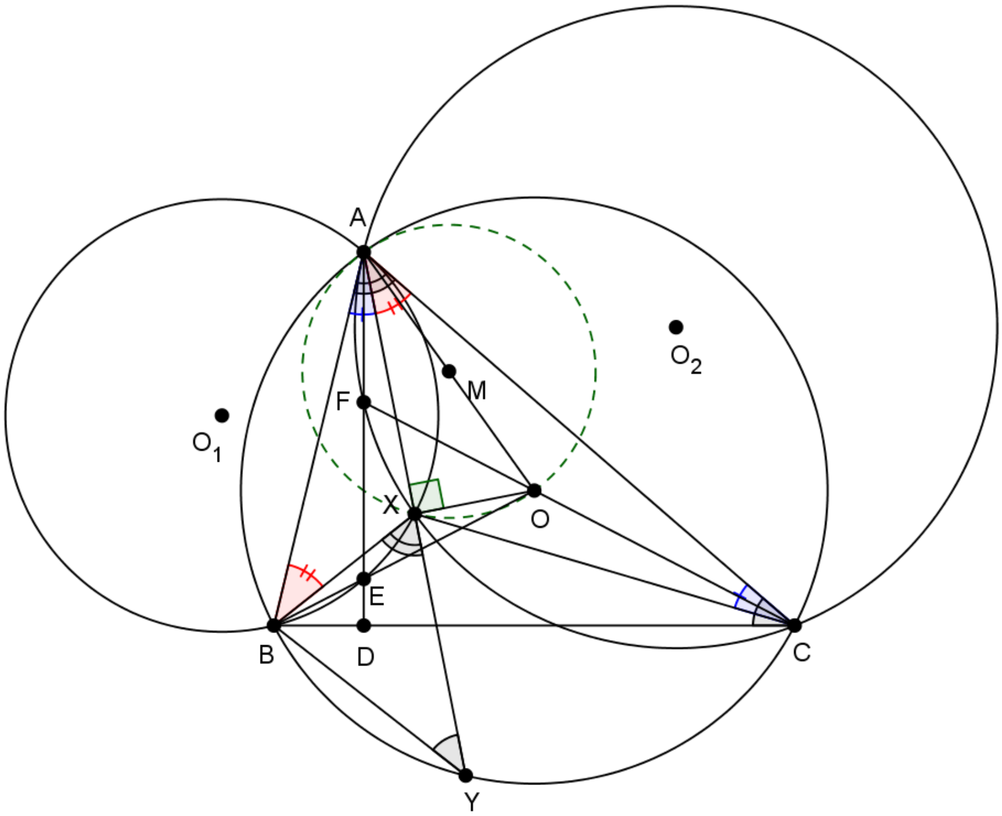



Let $\omega_1$ and $\omega_2$ be the circumcircles of $ABE$ and $ACF$, respectively.

$$\angle ACF \equiv \angle ACO = \frac{180° - \angle AOC}{2} = 90° - \beta = 90° - \angle ABD = \angle BAD \equiv \angle BAF$$

Therefore, $BA$ is tangent to $\omega_2$, i.e. $BA \perp AO_2$.
On the other hand, $\overline{O_1A} = \overline{O_1B}$ and $\overline{OA} = \overline{OB}$ as radii, so $O_1O \perp BA$.
Therefore, $AO_2 \parallel O_1O$. Similarly, $AO_1 \parallel O_2O$.
So, $AO_1OO_2$ is a parallelogram and because its diagonals bisect each other, $M \in O_1O_2$ ■

**Proof 2** (Stefan Lozanovski).




Let $\omega$, $\omega_1$ and $\omega_2$ be the circumcircles of $ABC$, $ABE$ and $ACF$, respectively. Let the second intersection of $\omega_1$ and $\omega_2$ be $X$.
Note that the points $O_1$ and $O_2$ and $M$ are the centers of $\omega_1$, $\omega_2$ and the circle with diameter $AO$. It is sufficient to prove that these three circles are coaxial, i.e. that $X$ lies on the circle with diameter $AO$.
We need to prove that $\angle AXO = 90°$, and because $O$ is the circumcenter of $ABC$, we need to prove that $X$ is the midpoint of the chord $AY$, where $Y$ is the second intersection of $AX$ and $\omega$.
Same as in Proof 1, we get that $BA$ is tangent to $\omega_2$ and $CA$ is tangent to $\omega_1$. From this, we get that $\angle XBA = \angle XAC$ and $\angle XAB = \angle XCA$, so $\triangle XBA \sim \triangle XAC$ and therefore

$$\frac{\overline{XB}}{\overline{XA}} = \frac{\overline{BA}}{\overline{AC}} \quad \dots (1)$$

Also, $\angle BXY = \angle BAX + \angle XBA = \angle BAX + \angle XAC = \angle BAC$ and $\angle BYX \equiv \angle BYA = \angle BCA$, so $\triangle BXY \sim \triangle BAC$ and therefore

$$\frac{\overline{BX}}{\overline{XY}} = \frac{\overline{BA}}{\overline{AC}} \quad \dots (2)$$

Finally, from $(1)$ and $(2)$ we can conclude that $\overline{AX} = \overline{XY}$ ■