Problem. Let the incircle of the triangle ABC touch the sides BC and CA at points D and E, respectively. The circle with diameter AB intersects the line DE at points G and H, such that $\overline{GE} < \overline{GD}$. Let ω be the circumcircle of ABC. The circumcircle of BDG intersects ω again at P. The circumcircle of AEH intersects ω again at Q. Let R be the intersection of the lines AP and BQ. Prove that the incenter I of the triangle ABC lies on the circumcircle of PQR.

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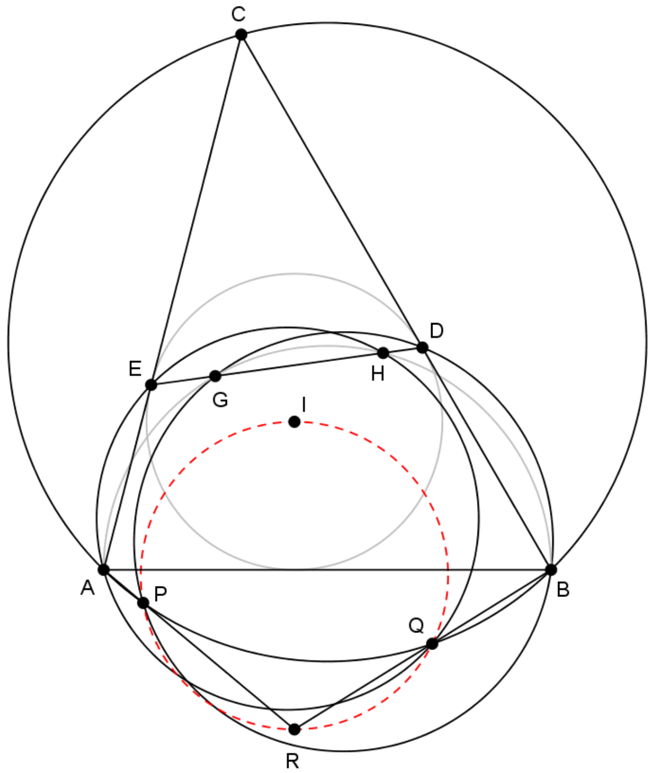


Proof (Stefan Lozanovski).
By Miquel's Theorem for the triangle EDC and the points $B \in DC$, $A \in CE$ and $G \in ED$ on its sides (or extensions), we get that the circumcircles (EAG), (DBG) and (CAB) pass through a common point. Therefore, P is the Miquel Point, so the circumcircle of EAG also passes through P. Similarly, the circumcircle of DBH passes through Q. We will prove that these two circles pass through I.
Let AI intersect DE at H'. Then,

$$\angle BIH' = \angle IAB + \angle IBA = \frac{\alpha + \beta}{2}.$$

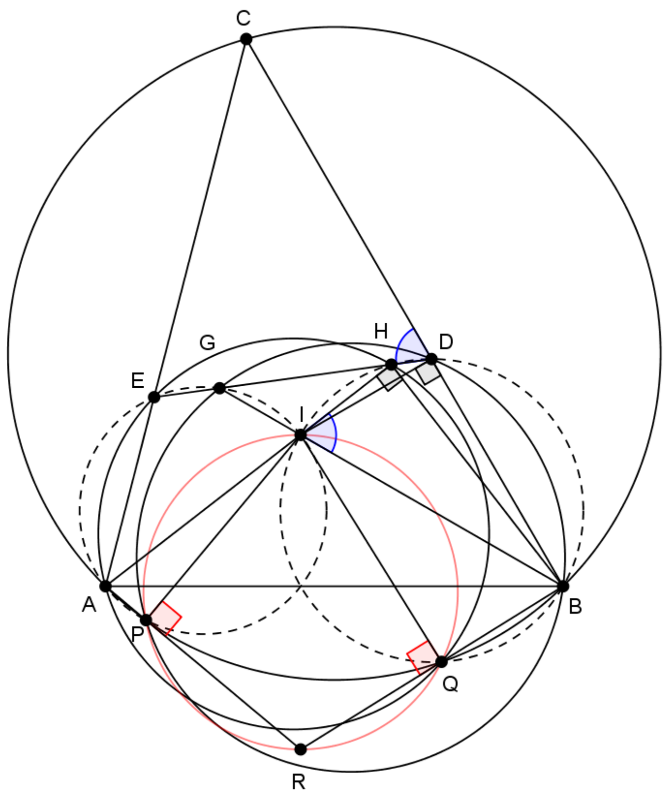
Since $\overline{CD} = \overline{CE}$ as tangent segments, we have

$$\angle CDH' \equiv \angle CDE = \frac{180° - \gamma}{2} = \frac{\alpha + \beta}{2},$$

so $BDH'I$ is cyclic. Therefore,

$$\angle AH'B \equiv \angle IH'B = \angle IDB = 90°,$$

so by the definition of H, we get that $H' \equiv H$ which means that $I \in AH$ and $I \in (BDH)$. Similarly, $I \in BG$ and $I \in (AEG)$.
In conclusion $BDHIQ$ is cyclic with diameter BI (because $\angle IDB = 90°$). So, $IQ \perp BR$. Similarly, $IP \perp AR$, and therefore $IPRQ$ is cyclic ■



Remark.
If an easier problem is needed, this problem can be slightly simplified by defining the points G and H as the intersections of the line DE with the angle bisectors of $\angle ABC$ and $\angle CAB$, respectively. Then, the points H' (and G') are not needed in this proof, but we still need the part where we prove that $BDHI$ (and $AEGI$) are cyclic.