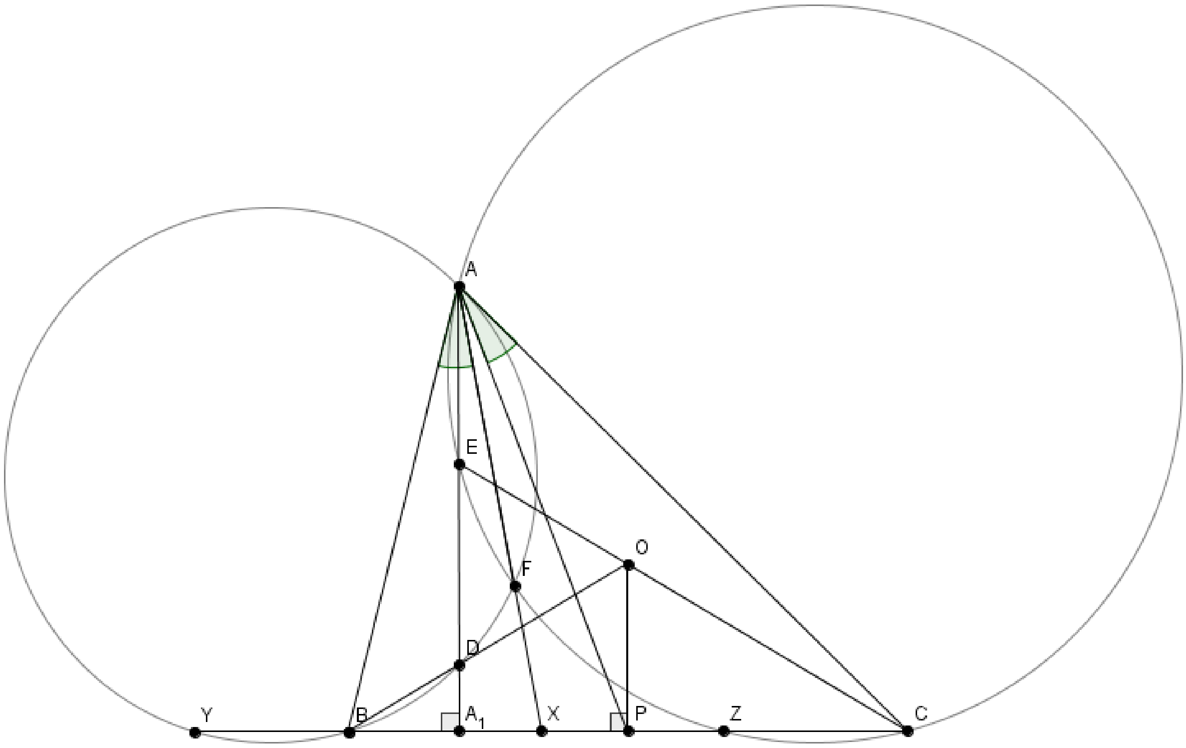
Problem: Let O be the circumcenter of the acute triangle ABC $\left(\overline{AB} < \overline{AC}\right)$. Let A_1 and P be the feet of the perpendiculars from A and O to BC, respectively. The intersections of the lines BO and CO with AA_1 are D and E, respectively. The circumcircles of the triangles ABD and ACE intersect again at F. Prove that the angle bisector of $\measuredangle FAP$ passes through the incenter of ABC.

Proposed by: Stefan Lozanovski, Macedonia

Solution 1: *(Stefan Lozanovski)*



We need to prove that $\measuredangle BAF = \measuredangle CAP$. Since OP is perpendicular to BC and O is the circumcenter, then P is the midpoint of BC. Since AP is the median from A, we need to prove that AF is the symmedian from A.

Let the line AF intersect the side BC at X and let the circumcircles of ABD and ACE meet the line BC again at Y and Z, respectively. Then, by the intersecting secant theorem, we have:

$$\overline{XB} \cdot \overline{XY} = \overline{XF} \cdot \overline{XA} = \overline{XZ} \cdot \overline{XC}$$

$$\frac{\overline{XB}}{\overline{XC}} = \frac{\overline{XZ}}{\overline{XY}} = \frac{\overline{XB} + \overline{XZ}}{\overline{XC} + \overline{XY}} = \frac{\overline{BZ}}{\overline{CY}} \quad \dots (1)$$

$$\measuredangle ACE \equiv \measuredangle ACO = \frac{1}{2}(180 - \measuredangle AOC) = \frac{1}{2}(180 - 2\measuredangle ABC) = 90 - \measuredangle ABC \equiv$$

$\equiv 90 - \measuredangle ABA_1 = \measuredangle BAA_1 \equiv \measuredangle BAE$, so BA is tangent to the circumcircle of ACE.

Similarly, CA is tangent to the circumcircle of ABD. By the tangent-secant theorem, we have:

$$\overline{BA}^2 = \overline{BZ} \cdot \overline{BC}$$

$$\overline{CA}^2 = \overline{CB} \cdot \overline{CY}$$

By dividing these two equations and using (1), we get:

$$\frac{\overline{BA}^2}{\overline{CA}^2} = \frac{\overline{BZ}}{\overline{CY}} = \frac{\overline{XB}}{\overline{XC}}$$

We proved that AX divides the side BC in the ratio of the squares of the sides AB and AC, so by *Lemma 1* we get that $AF \equiv AX$ is the A-symmedian in the triangle ABC ■

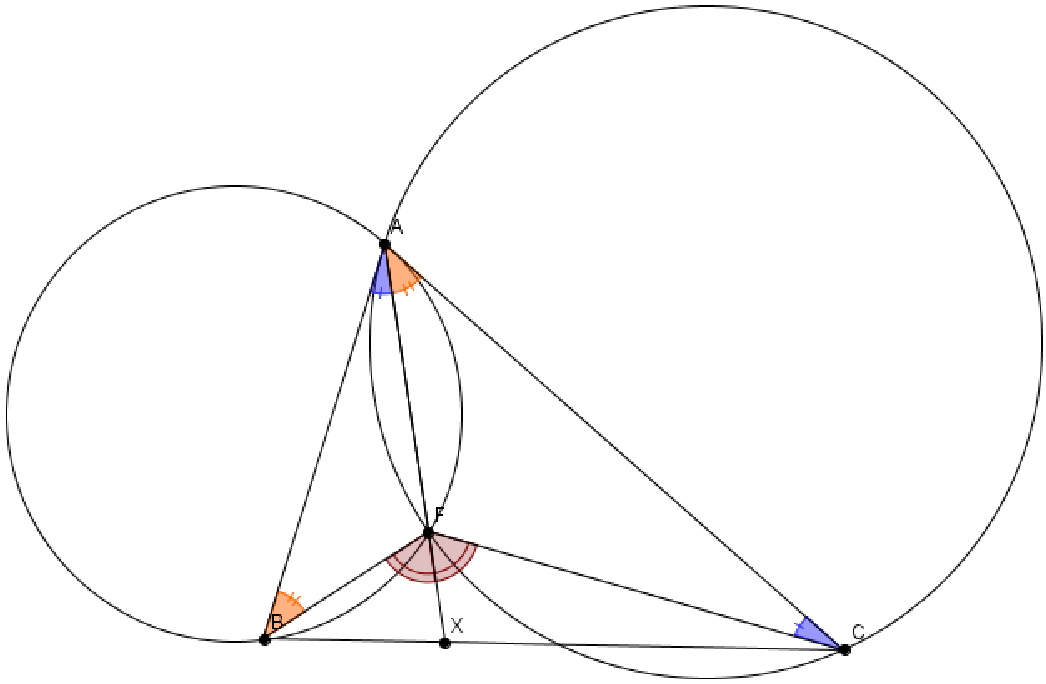
Solution 2: *(Stefan Lozanovski)*

We need to prove that $\measuredangle BAF = \measuredangle CAP$. Since *OP* is perpendicular to *BC* and *O* is the circumcenter, then *P* is the midpoint of *BC*. Since *AP* is the median from *A*, we need to prove that *AF* is the symmedian from *A*.

By some angle chasing:
$\measuredangle ACE \equiv \measuredangle ACO = \frac{1}{2}(180 - \measuredangle AOC) = \frac{1}{2}(180 - 2\measuredangle ABC) = 90 - \measuredangle ABC \equiv$
$\equiv 90 - \measuredangle ABA_1 = \measuredangle BAA_1 \equiv \measuredangle BAE$, we get that *BA* is tangent to the circumcircle of *ACF*. Similarly, *CA* is tangent to the circumcircle of *ABF*.

Now, we use the fact that the angle between a tangent and a chord is equal to any inscribed angle over the same chord:

$$\measuredangle BAF = \measuredangle ACF$$
$$\measuredangle ABF = \measuredangle CAF$$



So, the triangles *BAF* and *ACF* are similar which gives:

$$\frac{\overline{BF}}{\overline{CF}} = \frac{\overline{BF}/\overline{AF}}{\overline{CF}/\overline{AF}} = \frac{\overline{BA}/\overline{AC}}{\overline{AC}/\overline{AB}} = \frac{\overline{AB}^2}{\overline{AC}^2}$$

Also, $\measuredangle BFX = 180 - \measuredangle BFA = 180 - \measuredangle AFC = \measuredangle CFX$, so *FX* is an angle bisector in *BFC*, so:

$$\frac{\overline{BF}}{\overline{CF}} = \frac{\overline{BX}}{\overline{CX}}$$

From these two equalities, we get that

$$\frac{\overline{BX}}{\overline{CX}} = \frac{\overline{AB}^2}{\overline{AC}^2}$$

So, the line *AX* divides the side *BC* in the ratio of the squares of the sides *AB* and *AC*, so by *Lemma 1* we get that $AF \equiv AX$ is the symmedian from the vertex *A* in the triangle *ABC* ■

Solution 3: *(Stefan Lozanovski)*

We need to prove that $\measuredangle BAF = \measuredangle CAP$. Since *OP* is perpendicular to *BC* and *O* is the circumcenter, then *P* is the midpoint of *BC*. Since *AP* is the median from *A*, we need to prove that *AF* is the symmedian from *A*.

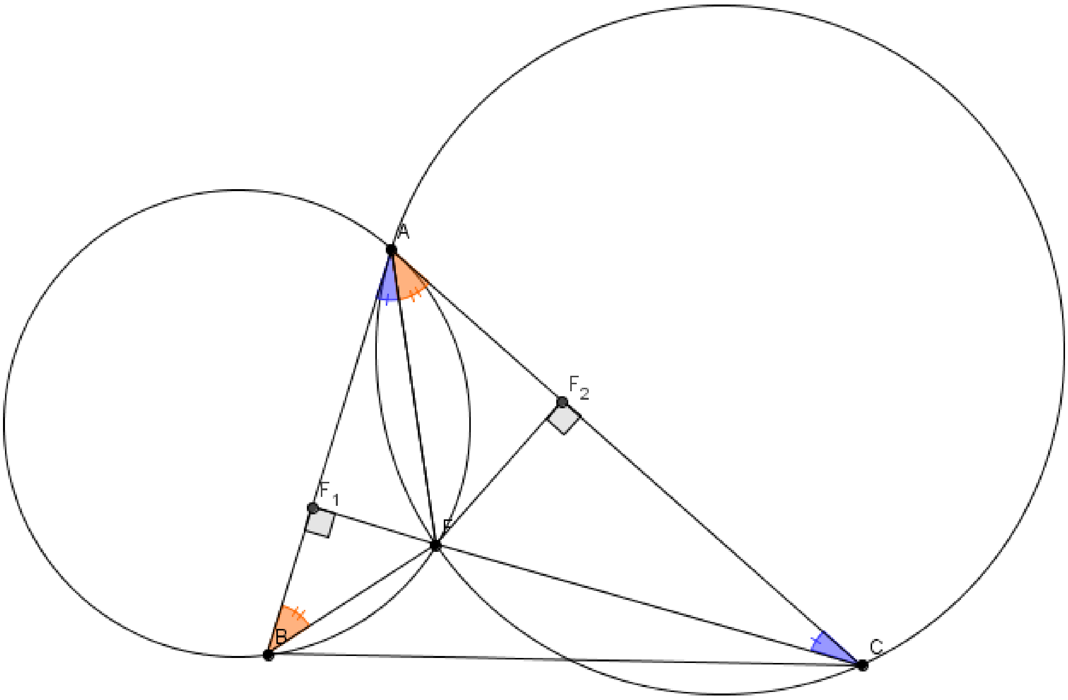
By some angle chasing:

$\measuredangle ACE \equiv \measuredangle ACO = \frac{1}{2}(180 - \measuredangle AOC) = \frac{1}{2}(180 - 2\measuredangle ABC) = 90 - \measuredangle ABC \equiv$
$\equiv 90 - \measuredangle ABA_1 = \measuredangle BAA_1 \equiv \measuredangle BAE$, we get that *BA* is tangent to the circumcircle of *ACF*. Similarly, *CA* is tangent to the circumcircle of *ABF*.

Now, we use the fact that the angle between a tangent and a chord is equal to any inscribed angle over the same chord:

$$\measuredangle BAF = \measuredangle ACF$$
$$\measuredangle ABF = \measuredangle CAF$$

So, the triangles *BAF* and *ACF* are similar.



Let F_1 and F_2 be the feet of the perpendiculars from *F* to the sides *AB* and *AC*, respectively. Then, from the similarity we have:

$$\frac{\overline{FF_1}}{\overline{FF_2}} = \frac{\overline{AB}}{\overline{AC}}$$

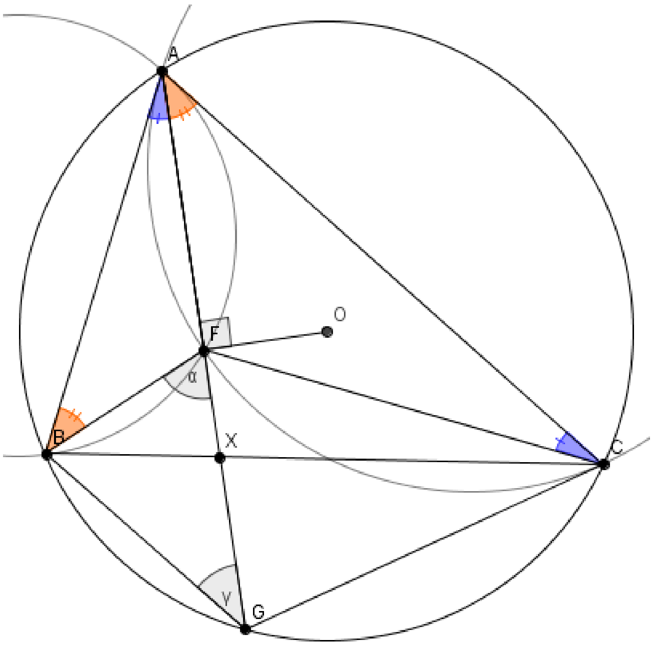
which means that the distances from *F* to the sides *AB* and *AC* are proportional to the lengths *AB* and *AC*, so by *Lemma 2b*, *F* lies on the symmedian from the vertex *A* in the triangle *ABC* ■

Solution 4: *(Stefan Lozanovski)*

We need to prove that $\measuredangle BAF = \measuredangle CAP$. Since *OP* is perpendicular to *BC* and *O* is the circumcenter, then *P* is the midpoint of *BC*. Since *AP* is the median from *A*, we need to prove that *AF* is the symmedian from *A*.

By some angle chasing:

$$\measuredangle ACE \equiv \measuredangle ACO = \frac{1}{2}(180 - \measuredangle AOC) = \frac{1}{2}(180 - 2\measuredangle ABC) = 90 - \measuredangle ABC \equiv$$



$\equiv 90 - \measuredangle ABA_1 = \measuredangle BAA_1 \equiv \measuredangle BAE$, we get that *BA* is tangent to the circumcircle of *ACF*.
Similarly, *CA* is tangent to the circumcircle of *ABF*.

Now, we use the fact that the angle between a tangent and a chord is equal to any inscribed angle over the same chord:

$$\measuredangle BAF = \measuredangle ACF$$
$$\measuredangle ABF = \measuredangle CAF$$

So, the triangles *BAF* and *ACF* are similar and:

$$\frac{\overline{BA}}{\overline{BF}} = \frac{\overline{AC}}{\overline{AF}} \ldots (1)$$

Let AX intersect the circumcircle of ABC again at G.

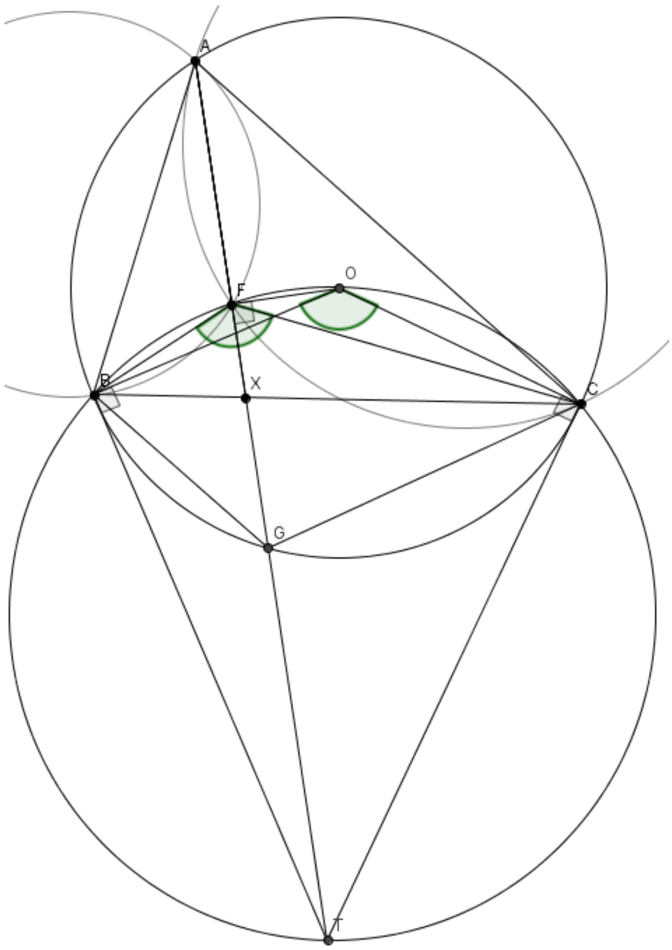
$$\measuredangle BFG = 180 - \measuredangle BFA = \measuredangle FBA + \measuredangle FAB =$$
$$= \measuredangle FAC + \measuredangle FAB = \measuredangle BAC = \alpha$$

$$\measuredangle BGF \equiv \measuredangle BGA = \measuredangle BCA = \gamma$$

So, the triangles *ABC* and *FBG* are also similar and:

$$\frac{\overline{AB}}{\overline{FB}} = \frac{\overline{AC}}{\overline{FG}} \ldots (2)$$

From (1) and (2) we get that $\overline{AF} = \overline{FG}$ and because *O* is the circumcenter, we get that $\measuredangle OFG = 90$.



Now, let's draw the tangents at *B* and *C* to the circumcircle of *ABC* and let them intersect at *T*. The quadrilateral *OBTC* is a cyclic quadrilateral with diameter *OT*.

Earlier in this solution, we proved that $\measuredangle BFG = \alpha$. Similarly, $\measuredangle CFG = \alpha$.
$\measuredangle BFC = \measuredangle BFG + \measuredangle CFG = \alpha + \alpha = 2\alpha = \measuredangle BOC$, so *F* lies on the circumcircle of *BOC* (with diameter *OT*). Because $\measuredangle OFG = 90$ and *OT* is the diameter of the circle, then *T* must lie on the line $FG \equiv AF$.

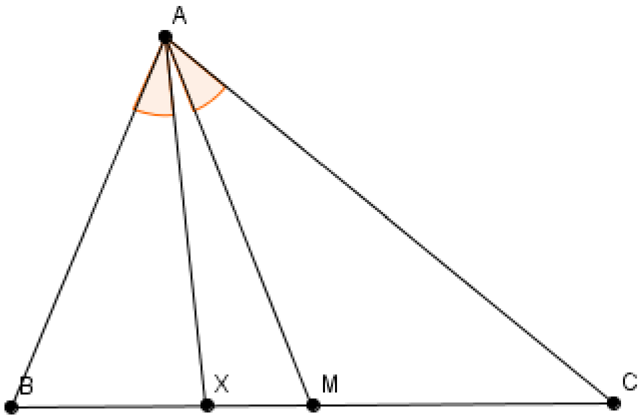
In conclusion, *AF* passes through the intersection of the tangents at *B* and *C* to the circumcircle of *ABC*, so by *Lemma 3b* we get that *AF* is the symmedian from the vertex *A* in the triangle *ABC* ■

Lemma 1: The line *AX* divides the opposite side *BC* in the ratio of the squares of the sides *AB* and *AC* if and only if *AX* is a symmedian in the triangle *ABC*.

$$\frac{\overline{BX}}{\overline{CX}} = \frac{\overline{AB}^2}{\overline{AC}^2}.$$

Proof:

Let *AM* and *AX*, be the median and symmedian from the vertex *A*, respectively.



$$\frac{\overline{BX}}{\overline{MC}} = \frac{Area(BAX)}{Area(MAC)} = \frac{\overline{BA} \cdot \overline{AX}}{\overline{AM} \cdot \overline{AC}}$$

$$\frac{\overline{BM}}{\overline{XC}} = \frac{Area(BMA)}{Area(CXA)} = \frac{\overline{BA} \cdot \overline{AM}}{\overline{AX} \cdot \overline{AC}}$$

By multiplying these equalities we get:

$$\frac{\overline{BX}}{\overline{CX}} = \frac{\overline{AB}^2}{\overline{AC}^2}$$

Since there is only one point on the line segment BC that divides it in a given ratio, the "only if" part is also true ■

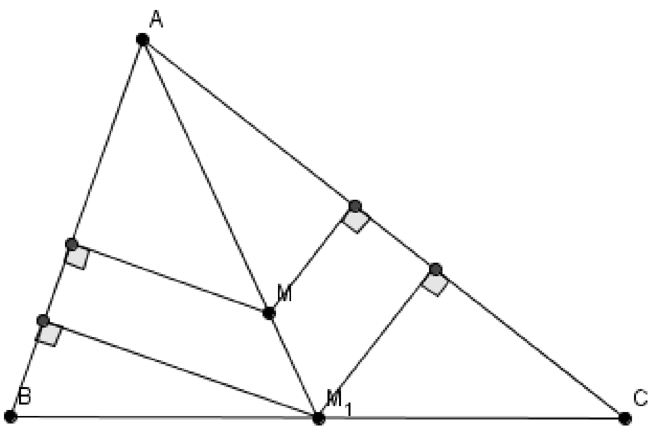
Lemma 2a: The *A*-median is the locus of the points *M* in the interior of $\measuredangle BAC$ such that

$$\frac{d(M,AB)}{d(M,AC)} = \frac{\overline{AC}}{\overline{AB}}.$$

Proof:

Let *M* be a point in the interior of $\measuredangle BAC$.

Let *AM* meet *BC* at M_1. Then,



$$\frac{d(M_1,AB)}{d(M_1,AC)} = \frac{d(M,AB)}{d(M,AC)} = \frac{\overline{AC}}{\overline{AB}}$$

$$\Leftrightarrow d(M_1,AB) \cdot \overline{AB} = d(M_1,AC) \cdot \overline{AC}$$

$$\Leftrightarrow Area(ABM_1) = Area(ACM_1)$$

$$\Leftrightarrow \overline{BM_1} = \overline{M_1C} \ ■$$

Lemma 2b: The *A*-symmedian is the locus of the points *L* in the interior of $\measuredangle BAC$ such that:

$$\frac{d(L,AB)}{d(L,AC)} = \frac{\overline{AB}}{\overline{AC}}.$$

Proof:

The symmedian is the reflection of the median with respect to the angle bisector, so by symmetry:

$$\frac{d(L,AB)}{d(L,AC)} = \frac{d(M,AC)}{d(M,AB)} = \frac{\overline{AB}}{\overline{AC}}$$

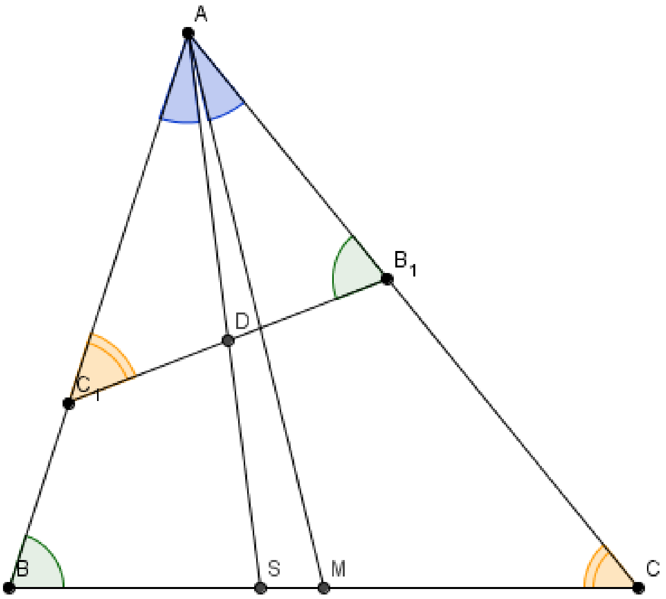
which means that the *A*-symmedian is the locus of the points *L* in the interior of $\measuredangle BAC$ such that:

$$\frac{d(L,AB)}{d(L,AC)} = \frac{\overline{AB}}{\overline{AC}} \ ■$$

<u>Lemma 3a:</u> A symmedian drawn from a vertex of a triangle divides the antiparallels to the opposite side in half.

Proof:
Let *AS* and *AM* be the symmedian and the median from the vertex *A*, respectively. Then, by the definition of symmedian, $\measuredangle BAS = \measuredangle CAM \ldots (1)$

Let *D* be the intersection of the lines *AS* and B_1C_1. By definition of antiparallel lines, the triangles *ABC* and $A_1B_1C_1$ are similar. Using (1) we get that the similarity maps *AM* to *AD*, so the symmedian *AS* passes through the midpoint of the side B_1C_1 which is antiparallel to *BC* (with respect to the lines *AB* and *AC)* ■



<u>Lemma 3b:</u> A symmedian through one of the vertices of a triangle passes through the point of intersection of the tangents to the circumcircle at the other two vertices.

Proof:
Let *BT* and *CT* be the tangents to the circumcircle of *ABC* at *B* and *C*. Then, because the angle between a tangent and a chord is equal to any inscribed angle over the same chord, $\measuredangle CBT = \measuredangle CAB = \alpha$ and $\measuredangle BCT = \measuredangle BAC = \alpha$, so the triangle *BCT* is isosceles and therefore $\overline{BT} = \overline{CT}$.

Let B_1C_1 be an antiparallel line to *BC* (with respect to the lines *AB* and *AC)* that passes through *T*. Then, $\measuredangle AB_1C_1 = \measuredangle ABC = \beta$. Now, $\measuredangle TCB_1 = 180 - \measuredangle ACB - \measuredangle BCT =$ $= 180 - \gamma - \alpha = \beta = \measuredangle AB_1C_1 \equiv \measuredangle CB_1T$, so the triangle TCB_1 is isosceles and therefore $\overline{B_1T} = \overline{CT}$. Similarly, $\overline{C_1T} = \overline{BT}$.

In conclusion, $\overline{C_1T} = \overline{BT} = \overline{CT} = \overline{B_1T}$, so *T* is the midpoint of B_1C_1. By *Lemma 3a*, *AT* is the symmedian from the vertex *A* ■